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On a Particular Construction of Skew-Selfadjoint Operator Matrices

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We consider a particular construction for skew-selfadjoint operator matrices, which are of central importance in initial boundary value problems of mathematical physics.

1 Introduction

Typical initial boundary value problems of mathematical physics can be represented in the general form

$$(\partial_0 \mathcal{M} + A)\, U = F, \tag{1}$$

where A is skew-selfadjoint, indeed commonly of the specific block matrix form

$$A = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \tag{2}$$

with $C : X_1 \subseteq X_0 \to Y$ a closed densely defined linear operator between Hilbert spaces X_0 and Y with $X_1 = D(C)$, see e.g. [4,5]. The operator $\mathcal{M}$ is referred to as the material law operator, which in the situation of interest here is a suitable linear operator acting on a Hilbert space realizing the proper space-time framework for the problem at hand.

The main purpose of this paper is to focus on the operator C in this construction of the skew-selfadjoint operator $A : D(C) \oplus D(C^*) \subseteq X_0 \oplus Y \to X_0 \oplus Y$ when Y is itself a direct sum of Hilbert spaces. In such a situation we shall loosely refer to a system of the form (1) as an abstract grad–div system. The guiding example, which at the same time motivates the name, is to take for C the differential operator ∇ with suitable domain X_1. The idea in this paper is to replace the role of the partial derivatives in ∇ by general operators in general Hilbert spaces, hence the term abstract grad-div systems for the corresponding evolutionary systems associated with the skew-selfadjoint operator A constructed according to (2). To illustrate the utility of the concept we consider a case of interest in connection with the boundary constraint equations such as the Leontovich boundary condition of electrodynamics, see e.g. [2,3,7].

2 Construction of Abstract grad–div Systems.

In this section, we shall reconsider the concept of the adjoint operator of a densely defined, closed linear operator C, specifically in order to deal with the fact that the image space Y of the operator C is given as an orthogonal sum of Hilbert spaces. Let us first provide a precise definition of what we would like to call an abstract grad-div-system.

Definition 2.1 Let $C : X_1 \subseteq X_0 \to Y$, be a densely defined, closed linear operator with domain X_1 between Hilbert spaces X_0, Y. We shall refer to a system of the form (1) with A generated via (2), as an *abstract* grad–div *system,* if Y given as a direct sum, i.e. $Y := \bigoplus_{k\in\{1,\ldots,n\}} Y_k$, for Hilbert spaces Y_k, $k \in \{1, \ldots, n\}$, $n \in \mathbb{N}$.

As a matter of jargon we shall say that the abstract grad–div system is *generated by* C. If ι_{Y_k} denotes the canonical isometric embedding of Y_k into Y then, with $C_k := \iota_{Y_k}^* C$, $k \in \{1, \ldots, n\}$, we have $Cx = C_0 x \oplus \cdots \oplus C_n x \equiv \begin{pmatrix} C_0 x \\ \vdots \\ C_n x \end{pmatrix} = \begin{pmatrix} C_0 \\ \vdots \\ C_n \end{pmatrix} x \in \begin{pmatrix} Y_0 \\ \vdots \\ Y_n \end{pmatrix} \equiv Y$ for $x \in X_1$. To clarify notation, we need the following definition.

Let $X_1 \hookrightarrow X_0 \hookrightarrow X_1'$ be a Gelfand triple. Consider a linear operator $S : X_1 \subseteq X_0 \to Y$ such that $L_S : X_1 \to Y,\ x \mapsto Sx$, is a continuous linear operator (S need not be closable) and define the operator $S^\diamond : Y \to X_1'$ by $\langle S^\diamond y | x\rangle_{X_0} = \langle y | Sx\rangle_Y$ for all $x \in X_1$, $y \in Y$, where here $\langle\,\cdot\,|\,\cdot\,\rangle_{X_0}$ denotes here the continuous extension of the inner product of X_0 to a duality pairing on $X_1' \times X_1$. With this concept of a dual we obtain the following result.

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Theorem 2.2 *Let C generate an abstract* grad–div *system with* $C = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$. *Then*

$C^* = \{((y_1, \dots, y_n), x) \in Y \oplus X_0 \mid x = \sum_{k=1}^n C_k^\diamond y_k \in X_0\}$.

3 An Application

We define the operator $\overset{\circ}{\operatorname{curl}}$ as the closure of the classical curl as an operator on $L^2(\Omega)^3$ with domain $\overset{\circ}{C}_\infty(\Omega)^3$, the space of smooth vector fields with compact support in $\mathbb{R}^3$.

By integration by parts we see $\overset{\circ}{\operatorname{curl}} \subseteq (\overset{\circ}{\operatorname{curl}})^* =: \operatorname{curl}$. We recall that $w \in D(\overset{\circ}{\operatorname{curl}})$ encodes $w|_{\partial\Omega} \times n = 0$ for domains Ω with sufficiently smooth boundary, where n denotes the unit outward normal vector field on $\partial\Omega$, see [1]. Let $L^2_\tau(\Gamma)$ denote the space of tangential vector-fields on Γ, i.e. $L^2_\tau(\Gamma) := \{f \in L^2(\Gamma)^3 \mid f \cdot n = 0\}$, which is a closed subspace of $L^2(\Gamma)^3$. Let π_τ be the continuous tangential component boundary trace operator $\pi_\tau : H(\operatorname{curl}, \Omega) \to V'_\gamma$ and let γ_τ be the continuous tangential boundary trace operator $\gamma_\tau : H(\operatorname{curl}, \Omega) \to V'_\pi$. Here V'_τ and V'_π are dual space of certain spaces V_τ and V_π, respectively, with $L^2_\tau(\Gamma)$ as pivot space for the two corresponding Gelfand triples, see [1] for details. We take for this example $X_0 := L^2(\Omega)^3$. Then A is constructed from $C = \begin{pmatrix} -\widetilde{\operatorname{curl}} \\ \tilde{\pi}_\tau \end{pmatrix} : X_1 \subseteq L^2(\Omega)^3 \to L^2(\Omega)^3 \oplus L^2_\tau(\Gamma)$, where $X_1 := \pi_\tau^{-1}\left[\operatorname{id}_\tau^\diamond\left[L^2_\tau(\Gamma)\right]\right]$ equipped with the graph norm of C is a Hilbert space, $\widetilde{\operatorname{curl}} := \operatorname{curl}|_{X_1} : X_1 \to L^2(\Omega)^3$ and $\tilde{\pi}_\tau := \pi_\tau|_{X_1} : X_1 \to L^2_\tau(\Gamma)$. For C, which is indeed a closed operator, to generate an abstract grad–div system the only thing left to show is that X_1 is dense in $L^2(\Omega)^3$. This, however, is trivial as $\overset{\circ}{C}_\infty(\Omega)^3 \subseteq X_1$. In physical terms the operator A acts on the triple (H, E, η), where H is the magnetic field, E the electric field and η represents, as we shall see, a quantity acting as the negative tangential boundary trace of E. To characterize containment in the domain of $\begin{pmatrix} -\widetilde{\operatorname{curl}} \\ \tilde{\pi}_\tau \end{pmatrix}^*$ we need a prerequisite:

Lemma 3.1 *Let $E \in D(\operatorname{curl})$, $\eta \in L^2_\tau(\Gamma)$. Then* $\operatorname{curl} E = \widetilde{\operatorname{curl}}^\diamond E - \tilde{\pi}_\tau^\diamond \eta$ *if and only if* $\gamma_\tau E + \eta = 0$ *on* $L^2_\tau(\Gamma)$.

Proof. We observe that for $\Psi \in H^1(\Omega)^3 \subseteq X_1$ the equation $(\gamma_\tau E + \eta)(\pi_\tau \Psi) = \langle \operatorname{curl} E | \Psi \rangle_{L^2(\Omega)^3} - \left(\widetilde{\operatorname{curl}}^\diamond E - \tilde{\pi}_\tau^\diamond \eta\right)(\Psi)$ holds true. Thus, if $\operatorname{curl} E = \widetilde{\operatorname{curl}}^\diamond E - \tilde{\pi}_\tau^\diamond \eta$, we get that $(\gamma_\tau E + \eta)(\pi_\tau \Psi) = 0$ for each $\Psi \in H^1(\Omega)^3$. Thus, $\gamma_\tau E + \eta = 0$ on $L^2_\tau(\Gamma)$, due to the density of $\tilde{\pi}_\tau[H^1(\Omega)^3] = V_\pi$ in $L^2_\tau(\Gamma)$, see [1, p. 850]. On the other hand, if $\gamma_\tau E + \eta = 0$, we immediately get $\operatorname{curl} E = \widetilde{\operatorname{curl}}^\diamond E - \tilde{\pi}_\tau^\diamond \eta$ by the density of $H^1(\Omega)^3$ in $L^2(\Omega)^3$. □

Theorem 3.2 *We have* $\begin{pmatrix} -\widetilde{\operatorname{curl}} \\ \tilde{\pi}_\tau \end{pmatrix}^* \subseteq \begin{pmatrix} -\operatorname{curl} & 0 \end{pmatrix}$ *and* $D\left(\begin{pmatrix} -\widetilde{\operatorname{curl}} \\ \tilde{\pi}_\tau \end{pmatrix}^*\right)$ *is given by the set*

$\{(E, \eta) \in D(\operatorname{curl}) \times L^2_\tau(\Gamma) \mid \gamma_\tau E + \eta = 0 \text{ on } L^2_\tau(\Gamma)\}$.

Proof. Note that with $\overset{\circ}{\operatorname{curl}} = \operatorname{curl}^*$ we have $\begin{pmatrix} -\overset{\circ}{\operatorname{curl}} \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} -\widetilde{\operatorname{curl}} \\ \tilde{\pi}_\tau \end{pmatrix} =: C$. From this we get $C^* \subseteq \begin{pmatrix} -\operatorname{curl} & 0 \end{pmatrix}$. Therefore, by Theorem 2.2, we obtain $(E, \eta) \in D(C^*)$ if and only if $E \in D(\operatorname{curl})$ and $-\operatorname{curl} E = -\widetilde{\operatorname{curl}}^\diamond E + \tilde{\pi}_\tau^\diamond \eta \in L^2(\Omega)^3$, which, in turn, by Lemma 3.1 is equivalent to $\gamma_\tau E + \eta = 0$ on $L^2_\tau(\Gamma)$ and $E \in D(\operatorname{curl})$. □

The latter theorem tells us that the containment in the domain of A gives – in the presence of a suitable material law – a boundary equation involving $\gamma_\tau E$ and $\tilde{\pi}_\tau H$. For a particular choice of material properties for example the Leontovich boundary condition can be recovered, for details see [6].

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